

AN ALTERNATIVE TO THE CHEVALLEY DESCRIPTION OF  $U[\mathfrak{sl}(n+1)]$  AND  $U_q[\mathfrak{sl}(n+1)]$ 

Tchavdar D. Palev\* and Preeti Parashar

International School for Advanced Studies, via Beirut 2- 4, 34013 Trieste, Italy

**Abstract.** An alternative to the Chevalley description of the universal enveloping algebra  $U[\mathfrak{sl}(n+1)]$  and of its  $q$ -deformed analogue  $U_q[\mathfrak{sl}(n+1)]$  in terms of generators and relations is given. In particular  $U[\mathfrak{sl}(n+1)]$  is an associative algebra with 1, generators  $\hat{a}_1^\pm, \dots, \hat{a}_n^\pm$  and relations  $[\hat{a}_1^\xi, \hat{a}_2^\xi] = 0$ ,  $[[\hat{a}_i^\xi, \hat{a}_j^{-\xi}], \hat{a}_k^\xi] = \delta_{jk} \hat{a}_i^\xi + \delta_{ij} \hat{a}_k^\xi$ ,  $|i - j| \leq 1$ ,  $\xi = \pm$ .

## 1. Introduction

In the present paper we describe firstly the special linear Lie algebra  $\mathfrak{sl}(n+1)$  and its universal enveloping algebra (UEA)  $U[\mathfrak{sl}(n+1)]$  via creation and annihilation generators (CAGs) [1]. Thus we provide an alternative to the Chevalley description of  $U[\mathfrak{sl}(n+1)]$  in terms of generators and relations. Secondly, we describe the quantized universal enveloping algebra  $U_q[\mathfrak{sl}(n+1)]$  [2, 3] entirely via deformed CAGs.

The concept of creation and annihilation operators (or generators) of a (semi)simple Lie (super)algebra of rank  $n$  was introduced in [1]. The annihilation generators  $\hat{a}_1^-, \dots, \hat{a}_n^-$  (resp. the creation generators  $\hat{a}_1^+, \dots, \hat{a}_n^+$ ) are always among the positive (resp. among the negative) root vectors. To be more precise we

---

\* Permanent Address: Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria; e-mail: tpalev@inrne.acad.bg

recall that the root system of  $sl(n+1) \equiv A_n$  is

$$\Delta = \{\varepsilon_A - \varepsilon_B | A \neq B = 0, 1, \dots, n\}. \quad (1)$$

The correspondence between the CAGs of  $sl(n+1)$  and their roots reads:  $\hat{a}_i^\pm \leftrightarrow \mp(\varepsilon_0 - \varepsilon_i)$ . The simple root vectors  $\hat{e}_i$  have roots  $\varepsilon_{i-1} - \varepsilon_i$ ,  $\hat{e}_i \leftrightarrow \varepsilon_{i-1} - \varepsilon_i$ ,  $i = 1, \dots, n$ . Therefore the CAGs are very different from the Chevalley generators  $\hat{e}_i, \hat{f}_i, \hat{h}_i$ ,  $i = 1, \dots, n$  of  $sl(n+1)$ .

For us personally the interest in the subject stems from the observation that the CAGs (including  $sl(\infty)$ ) provide an alternative way for quantization of integer spin fields. This leads to a generalization of the quantum statistics, the  $A$ -statistics [4]. The latter, as it is clear now, is a particular case of the Haldane exclusion statistics [5], a subject of considerable interest in condensed matter physics (see, for instance, [6-8] and the references therein). We are not going to discuss the properties of the underlying statistics here. We note only that the quantization of  $U[sl(n+1)]$  leads to deformations of the Fock space representations, considered in [4], which gives new solutions for the microscopic statistics of the  $g$ -ons of Karabali and Nair [9], a particular realization of the Haldane statistics. The very fact that we consider Hopf algebra deformations of the creation and the annihilation operators has an additional advantage: using the comultiplication one may construct new representations of  $\hat{a}_1^\pm, \hat{a}_2^\pm, \dots$  in any tensor product of state spaces, namely new solutions for the  $g$ -on statistics.

## 2. The Lie algebra $sl(n+1)$

In order to define the CAGs of  $sl(n+1)$  it is convenient to consider it as a subalgebra of the Lie algebra  $gl(n+1)$ . The  $UEA$  of  $gl(n+1)$  can be defined as an associative algebra with 1 of the indeterminates  $\{e_{AB} | A, B = 0, 1, \dots, n\}$  subject to the relations (below and throughout  $[x, y] = xy - yx$ )

$$[e_{AB}, e_{CD}] = \delta_{BC}e_{AD} - \delta_{AD}e_{CB}. \quad (2)$$

Then  $gl(n+1)$  is a subalgebra of the Lie algebra  $U[gl(n+1)]$  with generators  $e_{AB}$ ,  $A, B = 0, 1, \dots, n$  and commutation relations (2).

The Cartan subalgebra  $H'$  of  $gl(n+1)$  has a basis  $e_{00}, e_{11}, \dots, e_{nn}$ . Let  $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$  be the dual basis,  $\varepsilon_A(e_{BB}) = \delta_{AB}$ . The root vectors of both  $gl(n+1)$  and  $sl(n+1)$  are  $e_{AB}$ ,  $A \neq B = 0, 1, \dots, n$ . The root of each  $e_{AB}$  is  $\varepsilon_A - \varepsilon_B$ . Then

$$sl(n+1) = \text{span}\{e_{AB}, e_{AA} - e_{BB} | A \neq B = 0, 1, \dots, n\}. \quad (3)$$

The canonical description of  $sl(n+1)$ , which is appropriate for Hopf algebra deformations, is given in terms of its Chevalley generators

$$\hat{e}_i = e_{i-1,i}, \quad \hat{f}_i = e_{i,i-1}, \quad \hat{h}_i = e_{i-1,i-1} - e_{ii}, \quad i = 1, 2, \dots, n, \quad (4)$$

and the  $n \times n$  Cartan matrix  $\{\alpha_{ij}\}$  with entries  $\alpha_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i-1,j}$ :  $U[sl(n+1)]$  is an associative algebra with 1 of the Chevalley generators subject to the Cartan relations

$$[\hat{h}_i, \hat{h}_j] = 0, \quad [\hat{h}_i, \hat{e}_j] = \alpha_{ij}\hat{e}_j, \quad [\hat{h}_i, \hat{f}_j] = -\alpha_{ij}\hat{f}_j, \quad [\hat{e}_i, \hat{f}_j] = \delta_{ij}\hat{h}_i, \quad (5)$$

and the Serre relations

$$\begin{aligned} [\hat{e}_i, \hat{e}_j] &= 0, \quad [\hat{f}_i, \hat{f}_j] = 0, \quad |i-j| > 1, \\ [\hat{e}_i, [\hat{e}_i, \hat{e}_{i\pm 1}]] &= 0, \quad [\hat{f}_i, [\hat{f}_i, \hat{f}_{i\pm 1}]] = 0. \end{aligned} \quad (6)$$

The creation and the annihilation generators of  $sl(n+1)$  are [4]  $\hat{a}_i^+ = e_{i0}$  and  $\hat{a}_i^- = e_{0i}$ ,  $i = 1, \dots, n$ .

Our new result is contained in the following proposition.

*Proposition 1.*  $U[sl(n+1)]$  is an associative algebra with 1, (free) generators  $\hat{a}_1^\pm, \hat{a}_2^\pm, \dots, \hat{a}_n^\pm$  and relations

$$[\hat{a}_1^\xi, \hat{a}_2^\xi] = 0, \quad [[\hat{a}_i^\xi, \hat{a}_j^{-\xi}], \hat{a}_k^\xi] = \delta_{jk}\hat{a}_i^\xi + \delta_{ij}\hat{a}_k^\xi, \quad |i-j| \leq 1, \quad \xi = \pm, \quad i, j, k = 1, \dots, n. \quad (7)$$

We skip the proof, since it is a particular case of the proof of the Theorem in Sect.3, when  $q \rightarrow 1$ . The expressions for the Chevalley generators in terms of the CAGs read ( $i \neq 1$ ):

$$\begin{aligned} \hat{e}_1 &= \hat{a}_1^-, \quad \hat{e}_i = [\hat{a}_{i-1}^+, \hat{a}_i^-], \\ \hat{f}_1 &= \hat{a}_1^+, \quad \hat{f}_i = [\hat{a}_i^+, \hat{a}_{i-1}^-], \\ \hat{h}_1 &= [\hat{a}_1^-, \hat{a}_1^+], \quad \hat{h}_i = [\hat{a}_i^-, \hat{a}_i^+] - [\hat{a}_{i-1}^-, \hat{a}_{i-1}^+], \end{aligned} \quad (8)$$

with inverse relations

$$\begin{aligned} \hat{a}_1^- &= \hat{e}_1, \quad \hat{a}_i^- = [[[\dots [[\hat{e}_1, \hat{e}_2], \hat{e}_3] \dots], \hat{e}_{i-1}], \hat{e}_i], \\ \hat{a}_1^+ &= \hat{f}_1, \quad \hat{a}_i^+ = [\hat{f}_i, [\hat{f}_{i-1}, [\dots [\hat{f}_3, [\hat{f}_2, \hat{f}_1]] \dots]]]. \end{aligned} \quad (9)$$

Only from (5), (6) and (9) one derives all relations among the CAGs:

$$[[\hat{a}_i^\xi, \hat{a}_j^{-\xi}], \hat{a}_k^\xi] = \delta_{jk} \hat{a}_i^\xi + \delta_{ij} \hat{a}_k^\xi, \quad [\hat{a}_i^\xi, \hat{a}_j^\xi] = 0, \quad \xi = \pm, \quad i, j = 1, \dots, n. \quad (10)$$

Note that (7) is a subset of (10). This is to say that (7) represents the minimal set of relations required to define  $U[sl(n+1)]$ . The fact that the creation (annihilation) generators commute among themselves facilitates the construction of the Fock representations of  $sl(n+1)$  (also in the case  $n = \infty$ ) [4]. A similar problem for the parastatistics of an arbitrary order [10, 11] remains still unsolved: explicit expressions for the transformations of the Fock spaces do not exist.

### 3. $U_q[sl(n+1)]$ in terms of deformed creation and annihilation generators

The description of  $U_q[sl(n+1)]$  in terms of Chevalley generators is well known. See, for instance [2, 3], where all Hopf algebra operations are explicitly written. Here we write only the algebra operations.  $U_q[sl(n+1)]$  is an associative algebra with 1 of its Chevalley generators  $e_i, f_i, k_i = q^{\hat{h}_i}, \bar{k}_i \equiv k_i^{-1}, i = 1, \dots, n$ , subject to the Cartan relations ( $\bar{q} \equiv q^{-1}$ )

$$k_i \bar{k}_i = \bar{k}_i k_i = 1, \quad k_i k_j = k_j k_i, \quad (11a)$$

$$k_i e_j = q^{\alpha_{ij}} e_j k_i, \quad k_i f_j = q^{-\alpha_{ij}} f_j k_i, \quad (11b)$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i - \bar{k}_i}{q - \bar{q}}, \quad (11c)$$

and the Serre relations

$$[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad |i - j| \neq 1, \quad (12a)$$

$$[e_i, [e_i, e_{i\pm 1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i\pm 1}]_q]_{\bar{q}} = 0, \quad [f_i, [f_i, f_{i\pm 1}]_{\bar{q}}]_q = [f_i, [f_i, f_{i\pm 1}]_q]_{\bar{q}} = 0. \quad (12b)$$

Here and throughout we assume that  $q$  is not a root of 1.

Having in mind the expressions (9) we define the deformed CAGs as follows (below and throughout

$[a, b]_x = ab - xba$ ):

$$a_1^- = e_1, \quad a_i^- = [[[\dots [e_1, e_2]_{\bar{q}}, e_3]_{\bar{q}} \dots]_{\bar{q}}, e_{i-1}]_{\bar{q}}, e_i]_{\bar{q}} = [a_{i-1}^-, e_i]_{\bar{q}}, \quad i \neq 1, \quad (13a)$$

$$a_1^+ = f_1, \quad a_i^+ = [f_i, [f_{i-1}, [\dots [f_3, [f_2, f_1]_q]_q \dots]_q]_q = [f_i, a_{i-1}^+]_q, \quad i \neq 1. \quad (13b)$$

Note that Eqs. (11), (12) are invariant with respect to the linear antiinvolution  $(\ )^*$ , defined as

$$(e_i)^* = f_i, (k_i)^* = \bar{k}_i, (q)^* = \bar{q}, ((x)^*)^* = x, (ab)^* = (b)^*(a)^*. \quad (14a)$$

Therefore, if  $F, G \in U_q[sl(n+1)]$  and  $F = G$ , then also  $(F)^* = (G)^*$ . In particular,

$$(a_i^\pm)^* = a_i^\mp. \quad (14b)$$

*Proposition 2.* The following "mixed" relations hold ( $i \neq 1$ ):

$$\begin{aligned} (a) \quad [e_i, a_j^-]_{q^{\delta_{i-1,j}-\delta_{ij}}} &= -q\delta_{i-1,j}a_i^-, & (b) \quad [f_i, a_j^+]_{q^{\delta_{i-1,j}-\delta_{ij}}} &= \delta_{i-1,j}a_i^+, \\ (c) \quad [e_i, a_j^+] &= \delta_{ij}a_{i-1}^+\bar{k}_i, & (d) \quad [f_i, a_j^-] &= -\delta_{ij}k_i a_{i-1}^-, \end{aligned} \quad (15)$$

*Proof.* The proof is based on repeated use of identities like  $(\bar{x} = x^{-1})$

$$\text{If } [a, b] = 0, \text{ then } [[a, c]_q, b]_p = [a, [c, b]_p]_q; \quad (16a)$$

$$\text{If } [a, c] = 0, \text{ then } (x + \bar{x})[b, [a, [b, c]_x]_x] = [a, [b, [b, c]_x]_{\bar{x}}]_{x^2} - [[b, [b, a]_{\bar{x}}]_{\bar{x}}, c]_{x^2}. \quad (16b)$$

We begin with (15a).

(i) Let  $j < i - 1$ . Then from (12a) one immediately has  $[e_i, a_j^-] = 0$ .

(ii) Let  $j = i - 1$ . From (13a)  $[a_{i-1}^-, e_i]_{\bar{q}} = a_i^-$  and therefore  $[e_i, a_{i-1}^-]_q = -qa_i^-$ .

(iii) Let  $j = i$ . If  $i = 2$ ,  $[e_2, a_2^-]_{\bar{q}} = [e_2, [e_1, e_2]_{\bar{q}}]_{\bar{q}} = -\bar{q}[e_2, [e_2, e_1]_q]_{\bar{q}} = 0$  according to (12b). If  $i > 2$  set  $a_i^- = [[a_{i-2}^-, e_{i-1}]_{\bar{q}}, e_i]_{\bar{q}}$ . Then  $[e_i, a_i^-]_{\bar{q}} = [e_i, [[a_{i-2}^-, e_{i-1}]_{\bar{q}}, e_i]_{\bar{q}}]_{\bar{q}}$  and, since  $[e_i, a_{i-2}^-] = 0$ , using twice (16a), one has:  $[e_i, a_i^-]_{\bar{q}} = [e_i, [a_{i-2}^-, [e_{i-1}, e_i]_{\bar{q}}]_{\bar{q}}]_{\bar{q}} = [a_{i-2}^-, [e_i, [e_{i-1}, e_i]_{\bar{q}}]_{\bar{q}}]_{\bar{q}} = 0$ , since  $[e_i, [e_{i-1}, e_i]_{\bar{q}}]_{\bar{q}} = -\bar{q}[e_i, [e_i, e_{i-1}]_q]_{\bar{q}} = 0$  according to (12b). Thus,  $[e_i, a_i^-]_{\bar{q}} = 0$ .

(iv) Let  $i + 1 = j$ . From (16) one has at  $x = \bar{q}$

$$[e_i, [[e_{i-1}, e_i]_{\bar{q}}, e_{i+1}]_{\bar{q}}] = (q + \bar{q})^{-1}([e_{i-1}, [e_i, [e_i, e_{i+1}]_{\bar{q}}]_q]_{\bar{q}^2} - [e_i, [e_i, e_{i-1}]_{\bar{q}}]_q, e_{i+1}]_{\bar{q}^2}) = 0, \quad (17)$$

according to (12b). Therefore  $[e_2, a_3^-] = [e_2, [[e_1, e_2]_{\bar{q}}, e_3]_{\bar{q}}] = 0$ . If  $i > 2$ ,  $a_{i+1}^- = [[[a_{i-2}^-, e_{i-1}]_{\bar{q}}, e_i]_{\bar{q}}, e_{i+1}]_{\bar{q}}$  and since  $a_{i-2}^-$  commutes with  $e_i, e_{i+1}$ , from (16a) one obtains  $a_{i+1}^- = [a_{i-2}^-, [[e_{i-1}, e_i]_{\bar{q}}, e_{i+1}]_{\bar{q}}]_{\bar{q}}$ . Therefore,  $[e_i, a_{i+1}^-] = [e_i, [a_{i-2}^-, [[e_{i-1}, e_i]_{\bar{q}}, e_{i+1}]_{\bar{q}}]_{\bar{q}}]_{\bar{q}} = [a_{i-2}^-, [e_i, [[e_{i-1}, e_i]_{\bar{q}}, e_{i+1}]_{\bar{q}}]_{\bar{q}}]_{\bar{q}} = 0$ , according to (17).

(v) Let  $j > i + 1$ . Then  $a_j^- = [[\dots [a_{i+1}^-, e_{i+2}]_{\bar{q}}, e_{i+3}]_{\bar{q}} \dots]_{\bar{q}}, e_{j-1}]_{\bar{q}}, e_j]_{\bar{q}}$  and, since  $e_i$  commutes with  $e_{i+2}, e_{i+3}, \dots, e_j$ , according to (16a), and it commutes also with  $a_{i+1}^-$ , according to (iv), one has  $[e_i, a_j^-] = 0$ .

The unification of (i)-(v) yields (15a). Applying the antiinvolution (14) on both sides of (15a) one obtains (15b).

We pass to prove (15c).

(i) For  $i > j$ , (15c) is an immediate consequence of (11c) and (13b).

(ii) Let  $i = j$ .  $[e_i, a_i^+] = [e_i, [f_i, a_{i-1}^+]_q]$  (from (i) and (16a))  $= [[e_i, f_i], a_{i-1}^+]_q$   
 $= [(q - \bar{q})^{-1}(k_i - \bar{k}_i), a_{i-1}^+]_q = a_{i-1}^+ \bar{k}_i$ .

(iii) Let  $j = i + 1$ . Using (16a) and then (15a), we have

$$\begin{aligned} [e_i, a_{i+1}^+] &= [e_i, [f_{i+1}, [f_i, a_{i-1}^+]_q]_q] = [f_{i+1}, [[e_i, f_i], a_{i-1}^+]_q]_q \\ &= [f_{i+1}, [(q - \bar{q})(k_i - \bar{k}_i), a_{i-1}^+]_q]_q = [f_{i+1}, a_{i-1}^+ \bar{k}_i]_q = 0. \end{aligned}$$

(iv) For  $j > i + 1$ ,  $a_j^+ = [f_j, [f_{j-1}, [\dots [f_{i+3}, [f_{i+2}, a_{i+1}^+]_q] \dots]_q]_q$  and since  $e_i$  commutes with  $a_{i+1}^+$  according to (iii) and with  $f_j, f_{j-1}, \dots, f_{i+2}$  according to (11c), we conclude that  $[e_i, a_j^+] = 0$ .

Combining (i)-(iv) gives (15c). Acting on both sides of it with the antiinvolution (14) one derives (15d).

*Proposition 3.* The deformed CAGs (13) together with the "Cartan" operators  $k_1, \dots, k_n$  generate (in a sense of an associative algebra)  $U_q[sl(n+1)]$ .

*Proof.* The proof is an immediate consequence of the relations:

$$[a_i^-, a_i^+] = \frac{k_1 k_2 \dots k_i - \bar{k}_1 \bar{k}_2 \dots \bar{k}_i}{q - \bar{q}}, \quad i = 1, \dots, n, \quad (18a)$$

$$[a_i^-, a_{i+1}^+] = -k_1 k_2 \dots k_i f_{i+1}, \quad [a_{i+1}^-, a_i^+] = -e_{i+1} \bar{k}_1 \bar{k}_2 \dots \bar{k}_i, \quad i = 1, \dots, n-1. \quad (18b)$$

These equations are proved by induction on  $i$ . For  $i = 1$  (18a) holds. Let (18a) be true. Then from (13b), (16a) and (18a)

$$[a_i^-, a_{i+1}^+] = [a_i^-, [f_{i+1}, a_i^+]_q] = [f_{i+1}, [a_i^-, a_i^+]_q] = \frac{1}{q - \bar{q}} [f_{i+1}, k_1 k_2 \dots k_i - \bar{k}_1 \bar{k}_2 \dots \bar{k}_i]_q.$$

Using (11) the latter yields

$$[a_i^-, a_{i+1}^+] = -q f_{i+1} k_1 k_2 \dots k_i = -k_1 k_2 \dots k_i f_{i+1}.$$

Similarly one derives the other equation in (18b). The conclusion so far is that if (18a) holds, then also

Eqs. (18b) hold. Assuming this, compute  $[a_{i+1}^-, a_{i+1}^+] = [[a_i^-, e_{i+1}]_{\bar{q}}, a_{i+1}^+]_{\bar{q}}$ . Using the identity  $[[A, B]_x, C] = [[A, C], B]_x + [A, [B, C]]_x$ , one has

$$\begin{aligned} [a_{i+1}^-, a_{i+1}^+] &= [[a_i^-, a_{i+1}^+], e_{i+1}]_{\bar{q}} + [a_i^-, [e_{i+1}, a_{i+1}^+]]_{\bar{q}} \\ &= -[k_1 k_2 \dots k_i f_{i+1}, e_{i+1}]_{\bar{q}} + [a_i^-, a_i^+ \bar{k}_{i+1}]_{\bar{q}} \end{aligned}$$

$$\begin{aligned}
&= [e_{i+1}, f_{i+1}] k_1 k_2 \dots k_i + [a_i^-, a_i^+] \bar{k}_{i+1} \\
&= (q - \bar{q})^{-1} (k_1 k_2 \dots k_{i+1} - \bar{k}_1 \bar{k}_2 \dots \bar{k}_{i+1}).
\end{aligned}$$

Thus, if (18a) holds for a certain  $i$ , it holds for any  $i$ . Hence Eqs. (18a) and (18b) hold. Therefore  $(i = 1, 2, \dots, n-1)$ ,

$$e_1 = a_1^-, \quad f_1 = a_1^+, \quad e_{i+1} = -[a_{i+1}^-, a_i^+] k_1 k_2 \dots k_i, \quad f_{i+1} = -\bar{k}_1 \bar{k}_2 \dots \bar{k}_i [a_i^-, a_{i+1}^+]. \quad (19)$$

Hence  $a_1^\pm, \dots, a_n^\pm$  together with  $k_1^{\pm 1}, \dots, k_n^{\pm 1}$  generate  $U_q[sl(n+1)]$ , which completes the proof.

Define new “Cartan” generators

$$L_i = k_1 k_2 \dots k_i \Leftrightarrow k_1 = L_1, \quad k_i = L_i L_{i-1}^{-1}, \quad i \neq 1. \quad (20)$$

We are now ready to formulate our main result.

*Theorem.*  $U_q[sl(n+1)]$  is an associative algebra with unity, generators  $a_i^\pm$ ,  $L_i$ ,  $L_i^{-1} \equiv \bar{L}_i$ ,  $i = 1, \dots, n$  and relations

$$L_i \bar{L}_i = \bar{L}_i L_i = 1, \quad L_i L_j = L_j L_i, \quad (21a)$$

$$L_i a_j^\pm = q^{\mp(1+\delta_{ij})} a_j^\pm L_i, \quad (21b)$$

$$[a_i^-, a_i^+] = \frac{L_i - \bar{L}_i}{q - \bar{q}}, \quad (21c)$$

$$[[a_i^\eta, a_{i+\xi}^{-\eta}], a_j^\eta]_{q^{\xi(1+\delta_{ij})}} = \delta_{j,i+\xi} L_j^{-\xi\eta} a_i^\eta, \quad \xi, \eta = \pm \text{ or } \pm 1, \quad (21d)$$

$$[a_1^\eta, a_2^\eta]_q = 0, \quad \eta = \pm. \quad (21e)$$

*Proof.* Most of the preliminary results, necessary for the proof, are already obtained. Eq. (21a) is an immediate consequence of (11a) and the definition (20). From (9), (11b) and (20) one derives  $L_i a_j^+ = q \sum_{r=1}^i \sum_{s=1}^j \alpha_{rs} a_j^+ L_i$ . Then (21b) follows from the observation that  $\sum_{r=1}^i \sum_{s=1}^j \alpha_{rs} = 1 + \delta_{ij}$ . Inserting (20) in (18a) one obtains (21c). Replacing in (15)  $e_i$  and  $f_i$  with the right hand sides of (19) one derives all triple relations (21d). The nontrivial part is to write all of them in the compact form (21d). Eqs. (21e) coincide with two of the Serre relations (12b).

Thus, Eqs. (21) hold. It remains to prove that these relations are sufficient in order to derive any other relation in  $U_q[sl(n+1)]$ . To this end it suffices to show that the Cartan relations (11) and the Serre relations

(12) follow from Eqs. (21). The relations for the Chevalley generators are (19) and (20). Eq. (11a) follows straight from (21a), and (11b) is obtained by the repeated use of (21b).

The proof of Eq. (11c) is however not so simple. We consider in detail the more difficult case, namely when  $i, j \neq 1$ . Then  $[e_i, f_j] = [[a_{i-1}^+, a_i^-]L_{i-1}, \bar{L}_{j-1}[a_j^+, a_{j-1}^-]]$   
 $= [[a_{i-1}^+, a_i^-], \bar{L}_{j-1}][a_j^+, a_{j-1}^-]L_{i-1} + \bar{L}_{j-1}[[a_{i-1}^+, a_i^-], [a_j^+, a_{j-1}^-]]L_{i-1}$   
 $+ [a_{i-1}^+, a_i^-]\bar{L}_{j-1}[L_{i-1}, [a_j^+, a_{j-1}^-]]$  since  $[L_{i-1}, \bar{L}_{j-1}] = 0$ .

Let us shift all the  $L$  and  $\bar{L}$  to the left by making repeated use of Eq.(21b). So r.h.s. becomes

$q^{1-\delta_{i-1,j-1}+\delta_{i,j-1}}\bar{L}_{j-1}L_{i-1}([a_{i-1}^+, a_i^-][a_j^+, a_{j-1}^-] - q^{\delta_{i-1,j}-\delta_{i,j-1}}[a_j^+, a_{j-1}^-][a_{i-1}^+, a_i^-])$ , which can be written as

$$[e_i, f_j] = q^{1-\delta_{i-1,j-1}+\delta_{i,j-1}}\bar{L}_{j-1}L_{i-1}[[a_{i-1}^+, a_i^-], [a_j^+, a_{j-1}^-]]_{q^{\delta_{i-1,j}-\delta_{i,j-1}}}. \quad (22)$$

Now we use the identity

$$[A, [B, C]_x]_y = [[A, B]_z, C]_t + z[B, [A, C]_r]_s \quad \text{with conditions } x = zs, y = zr, t = zsr. \quad (23)$$

- (i) For  $i = j$  (22) reduces to  $[e_i, f_i] = [[a_{i-1}^+, a_i^-], [a_i^+, a_{i-1}^-]] =$   
 $[[[a_{i-1}^+, a_i^-], a_i^+]_q, a_{i-1}^-]_{\bar{q}} - q[a_i^+, [[a_i^-, a_{i-1}^+], a_{i-1}^-]_{\bar{q}}]_{\bar{q}} = [\bar{L}_i a_{i-1}^+, a_{i-1}^-]_{\bar{q}} - q[a_i^+, \bar{L}_{i-1} a_i^-]_{\bar{q}}$  (using (21d))  
 $= \bar{L}_i[a_{i-1}^+, a_{i-1}^-] - \bar{L}_{i-1}[a_i^+, a_i^-]$  (from (21b))  $= \frac{(L_i \bar{L}_{i-1} - L_{i-1} \bar{L}_i)}{q - \bar{q}} = \frac{k_i - \bar{k}_i}{q - \bar{q}}.$
- (ii) For  $|i - j| > 1$  (22) reduces to  $[e_i, f_j] = q\bar{L}_{j-1}L_{i-1}[[a_{i-1}^+, a_i^-], [a_j^+, a_{j-1}^-]]$   
 $= q\bar{L}_{j-1}L_{i-1}\{[[[a_{i-1}^+, a_i^-], a_j^+]_q, a_{j-1}^-]_{\bar{q}} - q[a_j^+, [[a_i^-, a_{i-1}^+], a_{j-1}^-]_{\bar{q}}]_{\bar{q}}\} = 0$  according to (21d).
- (iii) For  $j = i - 1$  (22) reduces to  $[e_i, f_{i-1}] = q\bar{L}_{i-2}L_{i-1}[[a_{i-1}^+, a_i^-], [a_{i-1}^+, a_{i-2}^-]]_q$   
 $= q\bar{L}_{i-2}L_{i-1}([[[a_{i-1}^+, a_i^-], a_{i-1}^+]_{q^2}, a_{i-2}^-]_{\bar{q}} - q^2[a_{i-1}^+, [[a_i^-, a_{i-1}^+], a_{i-2}^-]_{\bar{q}}]_{\bar{q}^2}) = 0$  from (21d).
- (iv) The case  $i = j - 1$  is similar to (iii).

Eq. (11c), corresponding to  $i = 1$  or to  $j = 1$ , can be proved in a similar manner. What remains to be proved next, are the Serre relations. Let us consider the first relation in Eq.(12a).

- (i)  $i \neq 1$ .  $[e_i, e_j] = [[a_{i-1}^+, a_i^-]L_{i-1}, [a_{j-1}^+, a_j^-]L_{j-1}]$   
 $= q^{\delta_{i-1,j}-\delta_{i-1,j-1}}[[a_{i-1}^+, a_i^-], [a_{j-1}^+, a_j^-]]_{q^{\delta_{i,j-1}-\delta_{i-1,j-1}}}L_{i-1}L_{j-1}$  using (21b)  
 $= ([[[a_{i-1}^+, a_i^-], a_{j-1}^+]_q, a_j^-]_{\bar{q}} + q[a_{j-1}^+, [[a_{i-1}^+, a_i^-], a_j^-]_{\bar{q}}]_{\bar{q}})L_{i-1}L_{j-1}$  (from (23) and the condition  $|i - j| \neq 1$ )  
 $= 0$  due to (21d).

Applying the antiinvolution (14) on the first relation in (12a), one obtains the second relation in (12a).



(ii)  $i = 1$ . Then  $[e_1, e_j] = [a_1^-, [a_{j-1}^+, a_j^-]L_{j-1}] = [a_1^-, [a_{j-1}^+, a_j^-]]_q L_{j-1}$   
 $= q[[a_j^-, a_{j-1}^+], a_1^-]_{\bar{q}} L_{j-1} = 0$  from (21d) since  $j > 2$ .

We shall now prove the other Serre relations (12b).

(i) Let  $i \neq 1$ . Then  $[e_i, [e_i, e_{i+1}]_{\bar{q}}]_q = [[a_{i-1}^+, a_i^-]L_{i-1}, [[a_{i-1}^+, a_i^-]L_{i-1}, [a_i^+, a_{i+1}^-]L_i]_{\bar{q}}]_q$

Let us first evaluate the inner commutator . So

$$\begin{aligned} [e_i, e_{i+1}]_{\bar{q}} &= [[a_{i-1}^+, a_i^-], [a_i^+, a_{i+1}^-]]L_i L_{i-1} \text{ using (21b)} \\ &= ([[[a_{i-1}^+, a_i^-], a_i^+]_q, a_{i+1}^-]_{\bar{q}} + q[a_i^+, [[a_{i-1}^+, a_i^-], a_{i+1}^-]_{\bar{q}}]_{\bar{q}})L_i L_{i-1} \text{ from (23)} \\ &= [\bar{L}_i a_{i-1}^+, a_{i+1}^-]_{\bar{q}} L_i L_{i-1} \text{ (due to the triple relations (21d)).} \end{aligned}$$

The full commutator reduces to

$$\begin{aligned} [[a_{i-1}^+, a_i^-]L_{i-1}, [\bar{L}_i a_{i-1}^+, a_{i+1}^-]_{\bar{q}} L_i L_{i-1}]_q &= -[[a_{i-1}^+, a_{i+1}^-], [a_{i-1}^+, a_i^-]]_{\bar{q}} L_{i-1}^2 \text{ (applying (21a) and (21b))} \\ &= (-[[[a_{i-1}^+, a_{i+1}^-], a_{i-1}^+]_{\bar{q}^2}, a_i^-]_q + \bar{q}^2[a_{i-1}^+, [[a_{i+1}^-, a_{i-1}^+], a_i^-]_{\bar{q}^2}]L_{i-1}^2) \text{ (using identity (23))} \\ &= 0 \text{ from (21d).} \end{aligned}$$

(ii) For  $i = 1$ ,  $[e_1, [e_1, e_2]_{\bar{q}}]_q = [a_1^-, [a_1^-, [a_1^+, a_2^-]L_1]_{\bar{q}}]_q$ .

Let's compute the inner commutator:

$$\begin{aligned} [e_1, e_2]_{\bar{q}} &= [a_1^-, [a_1^+, a_2^-]L_1]_{\bar{q}} = [a_1^-, [a_1^+, a_2^-]]_q L_1 \text{ from (21b)} \\ &= ([[a_1^-, a_1^+], a_2^-]_q + [a_1^+, [a_1^-, a_2^-]_q])L_1 = [[a_1^-, a_1^+], a_2^-]_q L_1 \text{ (due to (21e))} \\ &= \frac{1}{(q-\bar{q})}[L_1 - \bar{L}_1, a_2^-]_q L_1 = a_2^-. \end{aligned}$$

Therefore  $[e_1, [e_1, e_2]_{\bar{q}}]_q = [a_1^-, a_2^-]_q = 0$ .

The other Serre relations are proved in a similar manner. This completes the proof.

#### 4. Concluding remarks

We have shown that apart from the Chevalley definition, the universal enveloping algebra  $U[sl(n+1)]$  of the Lie algebra  $sl(n+1)$  and also its  $q$ -deformed analogue, the Hopf algebra  $U_q[sl(n+1)]$ , allow alternative descriptions in terms of generators and relations. The generators are the (deformed) creation and annihilation operators of  $sl(n+1)$ . In this respect the present investigation is along the line of the results obtained in [12, 13, 14]. In [12] the algebra  $U_q[so(2n+1)]$  was quantized via its CAGs. The latter are directly related

to the quantum statistics: in the nondeformed case the CAGs  $f_1^\pm, \dots, f_n^\pm$ , of  $so(2n+1)$  are para-Fermi operators [4]. A concept of deformed para-Bose operators  $b_1^\pm, \dots, b_n^\pm$ , was introduced in [13]; these operators provide an alternative description of the quantum superalgebra  $U_q[osp(1/2n)]$ . The quantization of all Lie superalgebras  $osp(2n+1/2m)$ , namely of all (super)algebras from the class  $B$  in the Kac classification [15], via both deformed para-Bose and deformed para-Fermi operators was carried out in [14]. Therefore it is not surprising that the (deformed) CAGs of  $sl(n+1)$  are related to new quantum statistics, the exclusion statistics of Haldane [5]. Clearly the present results can be extended first of all, to all superalgebras from the class  $A$ , i.e., all Lie superalgebras  $sl(n/m)$  and their  $q$ -deformed analogues. The CAGs of  $sl(1/n)$  were studied in [1] and later on it was shown that they describe noncanonical quantum systems with new, quite unconventional properties [16, 17]. Again the new statistics is an exclusion statistics [17]. It will be interesting to extend the present approach to all simple Lie algebras and even further to all basic Lie superalgebras.

We have not written here the explicit action of the comultiplication  $\Delta$ , the counit  $\varepsilon$  and the antipode  $S$  on the CAGs. The expressions follow immediately from the known transformations of the Chevalley generators under  $\Delta$ ,  $\varepsilon$  and  $S$  [2, 3] and the relations (13). For instance,

$$\Delta a_i^- = [[[\dots [[\Delta e_1, \Delta e_2]_{\bar{q}}, \Delta e_3]_{\bar{q}} \dots]_{\bar{q}}, \Delta e_{i-1}]_{\bar{q}}, \Delta e_i]_{\bar{q}}, \quad (24)$$

where  $\Delta e_i = e_i \otimes 1 + \bar{k}_i \otimes e_i$ . Then from (19) one obtains ( $i \neq 1$ )

$$\Delta e_1 = a_1^- \otimes 1 + \bar{L}_1 \otimes a_1^-, \quad \Delta e_i = [a_{i-1}^+, a_i^-] L_{i-1} \otimes 1 + L_i \bar{L}_{i-1} \otimes [a_{i-1}^+ a_i^-] L_{i-1}. \quad (25)$$

Inserting Eqs.(25) in (24) one obtains an "explicit" expression for  $\Delta a_i^-$  via the CAGs. This expression is however very involved for large  $i$ . Moreover, it is quite assymetrical for different  $a_i^\pm$ . For instance,

$$\Delta a_1^- = a_1^- \otimes 1 + \bar{L}_1 \otimes a_1^-, \quad (26)$$

whereas

$$\Delta a_2^- = a_2^- \otimes 1 + \bar{L}_2 \otimes a_2^- + (q - \bar{q})[a_1^+, a_2^-] \otimes a_1^-. \quad (27)$$

Therefore, replacing in (26) the index 1 with 2, one does not obtain  $\Delta a_2^-$ . One may try to define new, simpler and more symmetric expressions for the action of  $\Delta$ ,  $\varepsilon$  and  $S$  on the CAGs, using the available

multiparameter deformations of  $U_q[sl(n+1)]$  in the coalgebra sector [18] and fixing appropriately some of the parameters. This is the first open problem, which we would like to state.

The second problem is to construct the Fock representations of the relations (21), namely of the deformed CAGs of  $U_q[sl(n+1)]$ . This will lead directly to new solutions for the  $g$ -on statistics of Karabali and Nair [9]. To this end, one has to find as a first step, an analogue of the triple relations (10) in the deformed case. This is equivalent to writing down the "commutation relations" between (almost) all Cartan-Weyl generators, expressed via the CAGs (the expressions of the Cartan-Weyl generators via the Chevalley generators and the commutation relations they satisfy are known [19]). As a second step, using the Poincaré-Birkhoff-Witt theorem (following from the triple relations), one can construct the Fock representations of  $U_q[sl(n+1)]$ .

## Acknowledgements

T. Palev is grateful to Prof. C. Reina for kind hospitality at SISSA, and thanks Dr. N. I. Stoilova for constructive discussions. He was supported by the Grant  $\Phi - 416$  of the Bulgarian Foundation for Scientific Research.

## References

1. Palev T. D. 1980 *Journ. Math. Phys.* **21**, 1293.
2. Drinfeld V.G. 1986 *Quantum Groups*, Proc. Int. Congr. Math. vol. 1, 798, Berkeley.
3. Jimbo M. 1986 *Lett. Math. Phys.* **11**, 247.
4. Palev T. D. 1976 Thesis, Institute of Nuclear Research and Nuclear Energy, Sofia;  
1977 Preprint JINR E17-10550; 1979 *Czech. J. Phys B* **29**, 91;  
1980 *Rep. Math. Phys.* **18**, 117, 129.
5. Haldane F.D. 1991 *Phys. Rev. Lett.* **67**, 937.
6. Wu Y-Sh. 1994 *Phys. Rev. Lett.* **73**, 922.
7. Polychronakos A.P. 1996 *Phys. Lett. B* **365**, 202.
8. Ilinskaya A.V., Ilinsky K.N. and Gunn J.M.F. 1996 *Nucl. Phys. B* **458**, 562.

9. Karabali D. and Nair V. P. 1995 *Nucl. Phys.* **B 438**, 551.
10. Green H. S. 1993 *Phys. Rev.* **90**, 270.
11. Ohnuki Y. and Kamefuchi S. 1982 *Quantum Field Theory and Parastatistics*  
(Berlin: Springer).
12. Palev T. D. 1994 *Lett. Math. Phys.* **31** 151.
13. Palev T. D. 1993 *J. Phys. A: Math. Gen.* **26** L1111;  
Hadjiivanov L. K. 1993 *J. Math. Phys.* **34** 5476;  
Palev T. D. and Van der Jeugt J. 1995 *J. Phys. A: Math. Gen.* **28** 2605.
14. Palev T. D. 1996 *J. Phys. A: Math. Gen.* **29** L171;  
1996 A description of the quantum superalgebra  $U_q[osp(2n + 1/2m)]$  via Green  
generators, *Preprints* IC/96/127 and SISSA-119/96/FM, q-alg/9607030.
15. Kac V. G. 1978 *Lect. Notes Math.* **626** 597.
16. Palev T. D. 1982 *Journ. Math. Phys.* **23** 1778.  
Palev T. D. and Stoilova N. I. 1994 *J. Phys. A: Math. Gen.* **27** 977 and 7387.
17. Palev T D and Stoilova N I 1996 Many-body Wigner quantum systems  
*Preprint* IC/96/82 (1996) and hep-th/9606011.
18. Dobrev V. K. and Preeti Parashar 1993 *J. Phys. A: Math. Gen.* **26** 6991.
19. Hiroyuki Y. 1989 *Publ. RIMS Kyout Univ.* **25** 503.